xxx
Real Analysis I
HW 1
Due 09/??/2013

1.Let (X,d) be a metric space and let A, B be two nonempty subsets of X. Prove that if $A \cap B \neq \emptyset$, then the following inequality holds for the diameters δ:

$$\delta(A \cup B) \leq \delta(A) + \delta(B).$$

Prove also that the diameter of A is equal to the diameter of the closure. i.e., $\delta(A) = \delta(\overline{A})$

Recall that $\delta(A) = sup\{d(x,y) : x, y \in A\}$.

■

2. Let $0 < a < b$ and

$$f(x) = \begin{cases} 1 & \text{if } x \in [a,b] \cap \mathbb{Q} \\ 0 & \text{if } x \in [a,b] \text{ is irrational.} \end{cases}$$

Find the upper and lower Riemann integrals of $f(x)$ over $[a,b]$.

■

3.Consider the function

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1\ , \\ 0 & 1 < x \leq 2\ . \end{cases}$$

(a) What is $F(x) = \int_0^x f(t)dt$ on $[0,2]$?

(b) Is $F(x)$ continuous?

(c) Is $F'(x) = f(x)$?

■

4. Give examples to illustrate that

a) the pointwise limit of continuous (respectively, differentiable) functions is not necessarily continuous (respectively, differentiable).

b) the pointwise limit of integrable functions is not necessarily integrable.

■

5. Give examples to illustrate that

a) there exist differentiable functions f_n and f such that $f_n \to f$ pointwise on [0,1] but

$$\lim_{n\to\infty} f_n'(x) \neq \left(\lim_{n\to\infty} f_n(x)\right)' \qquad \text{when } x = 1,$$

b) there exist continuous functions f_n and f such that $f_n \to f$ pointwise on [0,1] but

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \left(\lim_{n\to\infty} f_n(x)\right) dx.$$

■

6. Show that there exists a continuous function defined on $\mathbb{R}$ that is nowhere differentiable by proving the following:

a) Let $g(x) = |x|$ if $x \in [-1, 1]$. Extend g to be periodic. Sketch g and the first few terms of the sum

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g(4^n x)$$

b) Use the Weierstrass M-test to show that f is continuous.

c) Prove that f is not differentiable at any point.

■

7.

a) Show that the Cantor set C can also be described as of all those real numbers in $[0,1]$ which have ternary expansions

$$C = \{x : \; x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad a_n = 0, 2 \;\}$$

b) Show that the length of C is equal to zero.

c) Show that Cantor set C can be put into one-to-one correspondence with the interval $[0,1]$ and that cardinality of C is continuum.

d) Show that Cantor set is totally disconnected and perfect.

■